MODIFICATION OF THE REGULARIZATION OF THE IDEAL PLASTICITY EQUATIONS IN THE AXIAL SYMMETRY CASE

M. Sh. Shtein

UDC 539.374

1. System of Equations of an Ideally Plastic Body. Let us consider a body of revolution in the coordinate system r, φ, z by considering the stress and displacement rate tensor components to be independent of the angle φ.

Here the equilibrium equations have the form

$$\partial\sigma_r/\partial r + \partial\tau_{rz}/\partial z + (\sigma_r - \sigma_\varphi)/r = 0,$$
$$\partial\tau_{rz}/\partial r + \partial\sigma_z/\partial z + \tau_{rz}/r = 0. \tag{1.1}$$

The displacement velocity vector components along the r, z axes are denoted by u, v, respectively. The associative law [1, 2] with Mises and Tresk plasticity conditions is taken as the flow law:

$$(\sigma_r-\sigma_\varphi)^2+(\sigma_\varphi-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\tau_{rz}^2-3/2=0; \tag{1.2}$$

$$(\sigma_r-\sigma_z-2\sigma_\varphi+2\varkappa)^2-(\sigma_r-\sigma_z)^2-4\tau_{rz}^2=0, \tag{1.3}$$
$$\varkappa=\begin{cases}1, & \text{if}\quad \sigma_1 \quad\text{or}\quad \sigma_2>\sigma_\varphi,\\ -1, & \text{if}\quad \sigma_1\quad\text{or}\quad \sigma_2<\sigma_\varphi,\end{cases}$$

σ_1, σ_2 are the principal stress tensor components in the r, z plane. Here and henceforth, all the stress components are referred to 2 τ_s. Conditions (1.3) correspond to the faces of the Tresk plasticity prism $|\sigma_1-\sigma_\varphi|=1,\ |\sigma_2-\sigma_\varphi|=1$.* Then the associative flow law with the plastic potential (1.2) yields

$$\partial u/\partial r=\lambda(2\sigma_r-\sigma_\varphi-\sigma_z),\ \partial v/\partial z=\lambda(2\sigma_z-\sigma_r-\sigma_\varphi),$$
$$\partial u/\partial z+\partial v/\partial r=6\lambda\tau_{rz},\ u/r=\lambda(2\sigma_\varphi-\sigma_z-\sigma_r). \tag{1.4}$$

Analogously for the Tresk case

$$\partial u/\partial r=\lambda(\sigma_z-\sigma_\varphi+\varkappa),\ \partial v/\partial z=\lambda(\sigma_r-\sigma_\varphi+\varkappa),$$
$$\partial u/\partial z+\partial v/\partial r=-2\lambda\tau_{rz},\ u/r=\lambda(2\sigma_\varphi-\sigma_r-\sigma_z-2\varkappa). \tag{1.5}$$

We find the circumferential stress φ and the factor λ from (1.2), (1.3) and the expressions for u/r in (1.4), (1.5):

For the Mises conditions

$$\sigma_\varphi=\frac{\sigma_r-\sigma_z}{2}+\xi\Delta_1,\quad \lambda=\frac{u}{r}\frac{1}{2\xi\Delta_1},$$
$$\Delta_1=\sqrt{\frac{3}{4}\left\{1-\left[(\sigma_r-\sigma_z)^2+4\tau_{rz}^2\right]\right\}},\ \xi=\begin{cases}1, & \sigma_\varphi>\sigma\\ -1, & \sigma_\varphi<\sigma\end{cases},\quad \sigma=\frac{\sigma_1+\sigma_2}{2}; \tag{1.6}$$

for the Tresk conditions

$$\sigma_\varphi=\frac{\sigma_r+\sigma_z}{2}+\frac{\eta}{2}\Delta_2-\varkappa,\quad \lambda=\frac{u}{r}\frac{\eta}{\Delta_2}, \tag{1.7}$$

*The faces $|\sigma_1-\sigma_2|=1$ are omitted from consideration since they result in the trivial case of $u\equiv 0$ [2].

Ternopol'. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 132-140, May-June, 1984. Original article submitted January 21, 1983.

 0021-8944/84/2503-0460$08.50

$$\Delta_2 = \sqrt{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2},$$

$$\varkappa = 1,\ \eta = 1,\ \text{if}\ \sigma_1 - \sigma_\varphi = 1,\ \varkappa = -1,\ \eta = 1,\ \text{if}\ \sigma_\varphi - \sigma_1 = 1, \tag{1.7}$$

$$\varkappa = 1,\ \eta = -1,\ \text{if}\ \sigma_2 - \sigma_\varphi = 1,\ \varkappa = -1,\ \eta = -1,\ \text{if}\ \sigma_\varphi - \sigma_2 = 1.$$

Moreover, let us introduce the Levy variables

$$\sigma_r = \sigma + \tau\cos 2\psi,\ \sigma_z = \sigma - \tau\cos 2\psi,\ \tau_{rz} = \tau\sin 2\psi;$$
$$\sigma = (\sigma_1 + \sigma_2)/2 = (\sigma_r + \sigma_z)/2,\ \tau = (\sigma_1 - \sigma_2)/2$$

(ψ is the angle between the first principal direction and the r axis in the r, z plane) and let us substitute the values of σ_φ and λ (1.6) and (1.7) into the equilibrium equations (1.1) and into (1.4) and (1.5), respectively. We afterwards obtain a quasilinear system of first-order equations to determine the five functions σ, ψ, τ, u, v after execution of certain manipulations, thus

$$\begin{gathered}
\frac{\partial\sigma}{\partial r} - 2\tau\left(\sin 2\psi\frac{\partial\psi}{\partial r} - \cos 2\psi\frac{\partial\psi}{\partial z}\right) + \cos 2\psi\frac{\partial\tau}{\partial r} + \sin 2\psi\frac{\partial\tau}{\partial z} = \frac{f_1}{r},\\
\frac{\partial\sigma}{\partial z} + 2\tau\left(\cos 2\psi\frac{\partial\psi}{\partial r} + \sin 2\psi\frac{\partial\psi}{\partial z}\right) + \sin 2\psi\frac{\partial\tau}{\partial r} - \cos 2\psi\frac{\partial\tau}{\partial z} = -\frac{\tau\sin 2\psi}{r},\\
\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} = \frac{f_3}{r},\quad \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} = -\frac{u}{r},\\
-\sin 2\psi\frac{\partial u}{\partial r} + \cos 2\psi\frac{\partial v}{\partial r} + \cos 2\psi\frac{\partial u}{\partial z} + \sin 2\psi\frac{\partial v}{\partial z} = 0,
\end{gathered} \tag{1.8}$$

where

$$f_1 = \begin{cases} -\tau\cos 2\psi + \xi\sqrt{\frac{3}{4}(1 - 4\tau^2)} & \text{for the Mises conditions,}\\ -\tau\cos 2\psi + \eta\tau - \varkappa & \text{for the Tresk conditions,}\end{cases} \tag{1.9}$$

$$f_3 = \begin{cases} -\dfrac{3u\tau\sin 2\psi}{\xi\sqrt{\frac{3}{4}(1 - 4\tau^2)}} & \text{for the Mises conditions,}\\ -\eta u\sin 2\psi & \text{for the Tresk conditions}\end{cases} \tag{1.10}$$

Therefore, the differential operators of the system (1.8), corresponding to the Mises and Tresk plasticity conditions, agree and differ only in the right sides of the first and third equations. It is easy to note that the third equation in the system (1.8) is the result of eliminating the factor λ from the last relationships (1.4), (1.5), the fourth equation is the incompressibility condition (the result of adding the first three expressions (1.4) or (1.5)), and the last equation of the system follows as the ratio between the difference of the first two and the fourth in (1.4) and (1.5), and is the condition for coaxiality of the stress and strain rate tensors. Exact self-similar solutions of the system (1.8) are obtained in [3].

2. Hyperbolic Regularization, Relationships on the Characteristics. Let us subject the system (1.8) to a characteristic analysis for which we write it in the matrix form

$$A\mathbf{t}_r + B\mathbf{t}_z = \mathbf{f},$$

$$\mathbf{t} = \begin{bmatrix} t_1\\ t_2\\ t_3\\ t_4\\ t_5\end{bmatrix} = \begin{bmatrix}\sigma\\ \psi\\ \tau\\ u\\ v\end{bmatrix},\quad \mathbf{f} = \begin{bmatrix} f_1/r\\ -\tau\sin 2\psi/r\\ f_3/r\\ -u/r\\ 0\end{bmatrix},\quad \mathbf{t}_r = \frac{\partial\mathbf{t}}{\partial r},\quad \mathbf{t}_z = \frac{\partial\mathbf{t}}{\partial z}, \tag{2.1}$$

$$A = \begin{bmatrix} 1 & -2\tau c & d & 0 & 0\\ 0 & 2\tau d & c & 0 & 0\\ 0 & 0 & \mu & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -c & d\end{bmatrix},\quad B = \begin{bmatrix} 0 & 2\tau d & c & 0 & 0\\ 1 & 2\tau c & d & 0 & 0\\ 0 & 0 & \nu & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & d & c\end{bmatrix},$$

$$c = \sin 2\psi,\ d = \cos 2\psi,\ \mu = \nu = 0,$$

from which it is seen that the matrices A, B are degenerate for any functions $\psi(r, z)$, $\tau(r, z)$ and the system (1.8) does not result in normal form. The corresponding characteristic form [4, 5] is identically zero for any directions $n(n_1, n_2)$ of the r, z plane

$$(n_1, n_2) = 2(\mu n_1 + \nu n_2)\left[\left(n_1^2 - n_2^2\right)\cos 2\psi + 2n_1 n_2 \sin 2\psi\right]^2 \equiv 0, \tag{2.2}$$

and the equations for the stresses and strain rates (1.8) in the axial symmetry case, which are a direct consequence of the associative flow law with the Mises and Tresk plasticity conditions, are not subject to classification (a system without type).*

Let us consider the possibility of regularizing the system (1.8) by considering that the latter should be hyperbolic with multiple characteristics (the stress field characteristics agree with the velocity field characteristics). Let $\mu \neq 0$ and $\nu \neq 0$; then

$$\left[\left(n_1^2 - n_2^2\right)\cos 2\psi + 2n_1 n_2 \sin 2\psi\right]^2 = 0 \text{ or } \left(\frac{n_1}{n_2}\right)^2 + 2\left(\frac{n_1}{n_2}\right)\operatorname{tg} 2\psi - 1 = 0, \tag{2.3}$$

from which

$$n_1/n_2 = -\operatorname{tg}(\psi - \pi/4), \ (n_1/n_2) = -\operatorname{tg}(\psi + \pi/4). \tag{2.4}$$

Let us introduce the continuously differentiable functions $\varphi^{\mp}(r, z) = \text{const}$, that determine smooth curves in the r, z plane. As the direction vectors $n^{\mp}(n_1^{\mp}, n_2^{\mp})$ we take $n^{\mp} = \operatorname{grad}\varphi^{\mp}(r, z)$. Then we will have respectively along the lines $\varphi^{\mp}(r, z) = \text{const}$

$$\begin{gathered} dr/dz = \operatorname{ctg}(\psi - \pi/4) = a_1 = a_3 = a_4 \ (\alpha\text{- line}), \\ dr/dz = \operatorname{ctg}(\psi + \pi/4) = a_2 = a_5 \ (\beta\text{- line}). \end{gathered} \tag{2.5}$$

Therefore, if it is assumed that $\mu, \nu \neq 0$, then duplicate characteristics defined by (2.5) correspond to the characteristic form (2.2). It is seen from (2.5) that the α-, β-lines are mutually orthogonal and agree with the lines of maximal tangential stress in the r, z plane, i.e., are slip lines. Let us note that (2.3) corresponds to the first, second, and fourth, fifth equations of the system (1.8). The factor $\mu n_1 + \nu n_2$ corresponds to the third equation. Let us demand that n_1/n_2 equal $-\tan(\psi - \pi/4)$ or $-\tan(\psi + \pi/4)$ as in (2.4), i.e.,

$$\begin{gathered} \mu n_1 + \nu n_2 = 0 \Rightarrow n_1/n_2 = -\nu/\mu = -\operatorname{tg}(\psi + \gamma\pi/4) \Rightarrow \\ \nu = \varepsilon, \ \mu = \varepsilon \operatorname{ctg}(\psi + \gamma\pi/4). \end{gathered} \tag{2.6}$$

Here ε is a small parameter, and γ takes on the value 1 or -1. Thus, regularization of the system (1.8) substantially reduces to the insertion of an additional small component into the left side of the third equation

$$\varepsilon\left[\operatorname{ctg}\left(\psi + \gamma\frac{\pi}{4}\right)\frac{\partial\tau}{\partial r} + \frac{\partial\tau}{\partial z}\right] + \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} = \frac{f_3}{r}. \tag{2.7}$$

Therefore, it is proposed to use (2.7) instead of the third equation in the system (1.8). Here (1.8) are transformed into a hyperbolic quasilinear system with multiple characteristics ($\gamma = -1 \Rightarrow \alpha$-line is triple, and the β-line is double; $\gamma = 1 \Rightarrow \alpha$-line is double, the β-line is triple).

It can be noted that (2.7) allows of physical interpretation. In fact, according to (1.4) and (1.5), the shear velocity γ_{rz} at an arbitrary point of the plastic flow domain is proportional to the tangential stress τ_{rz} at this point, which is reflected by the right side

*It is shown in [6] that the axisymmetric problem with the Mises condition is elliptic, where the passage to a second order system is realized by differentiation. The equivalence of a system obtained thus and the original was not investigated, while such a passage is not trivial [4] for nonlinear systems, and not legitimate in the general case. It is only emphasized in [1] that the axisymmetric problem with Mises condition is not hyperbolic. Equations for the velocities (kinetically determinate problem) were investigated for the Tresk conditions in [2]. They turn out to be hyperbolic. If $\sin 2\psi$, $\cos 2\psi$ are expressed from the third equation in (1.8) and substituted into the fifth equation, then we arrive at the system obtained in [2].

of the equality (2.7). The derivative of the maximum tangential stress τ in the direction of one of the slip line families is written in the square brackets on the left. Therefore, if the existence of a certain system of slip lines with discretization parameter ε [7] is assumed in the plastic domain, then the shear γ_{rz} at each point is determined by the tangential stress τ_{rz} and the projection of the gradient of the maximal tangential stress τ with minus sign on the active family of slip lines since the elements are deformed along slip lines in the direction of greatest growth of the maximal tangential stress.

For definiteness, we consider that $\gamma = -1$ in (2.7), and we introduce the differentiation operators along the characteristics

$$\frac{d^-}{dz} = \frac{\partial}{\partial z} + \operatorname{ctg}\left(\psi - \frac{\pi}{4}\right)\frac{\partial}{\partial r} \quad \text{(along } \alpha\text{-line)}$$
$$\frac{d^+}{dz} = \frac{\partial}{\partial z} + \operatorname{ctg}\left(\psi + \frac{\pi}{4}\right)\frac{\partial}{\partial r} \quad \text{(along } \beta\text{-line)}$$

Then according to the algorithm of reduction of the system (1.8) to characteristic form [5] with (2.7) taken into account, we arrive at relations on the characteristics

$$\begin{gathered} d^-\sigma - 2\tau d^-\psi + k_1 d^- v = G_1 dr, \\ d^+\sigma + 2\tau d^+\psi + k_2 d^+ v = G_2 dr, \quad d^-\tau + k_3 d^- v = G_3 dr; \end{gathered} \tag{2.8}$$

$$d^- U - V d^-\psi = (a_1 U - V)dz/2r, \quad d^+ V + U d^+\psi = (a_5 V + U)dz/2r. \tag{2.9}$$

Here

$$G_1 = \frac{1}{r}\left[f_1 - \frac{\tau \sin 2\psi}{a_1} - \frac{1}{\varepsilon}(-f_3 + u \operatorname{tg} 2\psi)\right],$$
$$G_2 = \frac{1}{r}\left[f_1 - \frac{\tau \sin 2\psi}{a_2} - \frac{1}{\varepsilon} f_3\right], \quad G_3 = \frac{1}{\varepsilon r}(-f_3 + u \operatorname{tg} 2\psi),$$
$$k_1 = \frac{2a_1}{\varepsilon} \operatorname{tg} 2\psi, \quad k_2 = -\frac{2a_2}{\varepsilon} \operatorname{tg} 2\psi, \quad k_3 = \frac{2}{\varepsilon} \operatorname{tg} 2\psi,$$

U, V are the displacement velocities along the α-, β-lines, respectively, and f_1, f_3 are defined according to the Mises and Tresk conditions from (1.9) and (1.10). Let us note that as $r, \varepsilon \to \infty$ the equations (2.8) go over into the Hencke relationships, while (2.9) go over into the Gehringer plane problem of the theory of ideal plasticity.

3. Iteration Approach to the Solution of Boundary-Value Problems for the Systems (2.8), (2.9). The mathematical theory of quasilinear hyperbolic first order systems [4, 5] assures the existence, uniqueness, and correctness of the Cauchy problem, the characteristic and mixed problems for such systems. However, the solution of real applied problems for the systems (2.8) and (2.9) is difficult because the boundary conditions on the boundary surfaces for the systems under consideration should simultaneously contain stress (vector) and displacement velocity components and the boundary of the rigid and plastic domains is not known in advance. The joint assignment of the stress vectors and displacement velocities on the surface boundaries is not possible in real problems. Consequently, it would be desirable to have an algorithm to solve the boundary-value problems for the systems (2.8), (2.9) that would permit seeking the stress and velocity separately (by analogy with the plane strain problem) but in a definite sequence [8, 9]. For instance, we find U, V from (2.9) for a known function ψ, and then by substituting them as knowns in (2.8), we determine σ, ψ, τ, etc. The existence of such an approach is assured by the agreement of the domains of definiteness of the solutions for the stresses and the displacement velocities (the characteristics (2.5) agree).

In conformity with the above, we define the vector-functions **S**, **U** corresponding to the stress and strain state in the form

$$\mathbf{S} = [S_1, S_2, S_3, 0, 0], \quad \mathbf{U} = [0, 0, 0, u_4, u_5], \tag{3.1}$$

where

$$S_1 = \sigma = t_1, \; S_2 = \psi = t_2, \; S_3 = \tau = t_3, \; u_4 = u = t_4, \; u_5 = v = t_5.$$

Then t from (2.1) is represented by a direct sum and the system (2.8), (2.9) can be written as

$$\mathbf{l}_k\cdot\left[\frac{\partial(\mathbf{S}+\mathbf{U})}{\partial z}+a_k\frac{\partial(\mathbf{S}+\mathbf{U})}{\partial r}\right]=g_k(r,\mathbf{S}+\mathbf{U})\quad(k=1,2,3);\tag{3.2}$$

$$\mathbf{l}_k\cdot\left[\frac{\partial\mathbf{U}}{\partial z}+a_k\frac{\partial\mathbf{U}}{\partial r}\right]=g_k(r,\mathbf{S}+\mathbf{U})\quad(k=4,5),\tag{3.3}$$

where g_k are the right sides of (2.2) and (2.9), while $\mathbf{l}_k$ are the eigenvectors of the characteristic matrix of the system (1.8), (2.7) corresponding to the eigennumbers a_k:

$$\mathbf{l}_1=[1,-2S_3,0,0,k_1],\ \mathbf{l}_2=[1,2S_3,0,0,k_2],$$
$$\mathbf{l}_3=[0,0,1,0,-k_3],\ \mathbf{l}_4=[0,0,0,a_4,1],\ \mathbf{l}_5=[0,0,0,a_5,1].$$

We shall consider the initial conditions of the Cauchy problem for the system (1.8) with the regularization (2.7) to be given on the segment $a\leq r\leq b$ of the $z=0$ axis (the general Cauchy problem reduces to that under consideration by replacement of the independent variables r, z, which does not alter the form of the equation (1.8)):

$$\sigma(r,0)=\sigma_0(r),\ \psi(r,0)=\psi_0(r),\ \tau(r,0)=\tau_0(r),$$
$$u(r,0)=u_0(r),\ v(r,0)=v_0(r).$$

According to the vector-functions (3.1) introduced, this corresponds to the following initial conditions for the characteristic system (3.2), (3.3)

$$\mathbf{S}_0(r)=\left[S_1^0(r),S_2^0(r),S_3^0(r),0,0\right],\quad \mathbf{U}_0(r)=\left[0,0,0,u_4^0(r),u_5^0(r)\right].\tag{3.4}$$

It is assumed that $\mathbf{S}_0(r),\mathbf{U}_0(r)\in G^1_{[a,b]}$. The smoothness of the remaining initial data g_k, k_k, a_k follows from (1.9), (1.10), (2.5), (2.8), (2.9) for $r,\varepsilon>0$.

We determine an iteration process for the construction of the solution of the problem (3.2)-(3.4). Let $U^{(1)}$ be a vector-function belonging to $C^{(1)}$ and such that $U^{(1)}(r,0)=U_0(r)$. Substituting $U^{(1)}(r,z)$ in the system (3.2), we determine $S^{(1)}$ as the solution of the Cauchy problem with initial conditions $S^{(1)}(r,0)=S_0(r)$. Afterwards, the solution $S^{(1)}(r,z)$ that has been found is substituted into (3.3) and the Cauchy problem $U^{(2)}(r,0)=U_0(r)$ is solved for a linear system. Then the definite approximation $U^{(2)}(r,z)$ is substituted into (3.2) and $S^{(2)}(r,0)=S_0(r)$ is determined. And thus this process is repeated over and over.

Let the approximation $U^{(i)}\in C^1$ be constructed; then we have for $S^{(i)}$

$$\mathbf{l}_k^{(i)}\cdot\left[\frac{\partial\mathbf{S}^{(i)}}{\partial z}+a_k^{(i)}\frac{\partial\mathbf{S}^{(i)}}{\partial r}\right]=g_k^{(i)}-\mathbf{l}_k^{(i)}\cdot\left[\frac{\partial\mathbf{U}^{(i)}}{\partial z}+a_k^{(i)}\frac{\partial\mathbf{U}^{(i)}}{\partial r}\right]\quad(k=1,2,3),\tag{3.5}$$
$$\mathbf{S}^{(i)}(r,0)=\mathbf{S}_0(r),$$

and we determine $U^{(i+1)}$ from the solution of the Cauchy problem for a linear system

$$\mathbf{l}_k^{(i)}\cdot\left[\frac{\partial\mathbf{U}^{(i+1)}}{\partial z}+a_k^{(i)}\frac{\partial\mathbf{U}^{(i+1)}}{\partial r}\right]=g_k^{(i)}\quad(k=4,5),\quad \mathbf{U}^{(i+1)}(r,0)=\mathbf{U}_0(r).\tag{3.6}$$

It follows from the existence theorem for the solution for quasilinear and linear systems [3, 4] that a solution $S^{(i)}$, $U^{(i+1)}\in C^{(1)}$ exists in the domain of definiteness $G^{(i)}$ of the problems (3-5), (3.6). So that all the approximations are determined and continuously differentiable in the domains $G^{(i)}$ (the domains of definiteness of the problems (3.5), (3.6) agree since the characteristics of the system (3.5) are also the characteristics of the system (3.6)). Moreover, since the initial problem (3.2)-(3.4) is quasilinear, then the domain G of definiteness of its solution is sought simultaneously with the solution S, U and is not generally known in advance. According to [4, 5], a domain $G_0\subseteq G$ of the variables r, z can be indicated in which the solution and its first derivative remain known to be bounded.

4. Boundedness of the Successive Approximations and the First Derivatives. Let us show that there exists a certain domain G_1 belonging to the domain G_0 and all domains $G^{(i)}$ such that the boundedness of $S^{(i)}$, $U^{(i)}$ and their first derivatives holds therein. To do this, we write the continued system for (3.2) and (3.3). The continued system for quasilinear hyperbolic equations is determined by differentiating the initial equation with respect to the independent variable [5] and results in the Riemann invariants which have the following form in our case

$$T_k = \mathbf{l}_k \cdot \frac{\partial (\mathbf{S} + \mathbf{U})}{\partial r} \quad (k = 1, 2, \ldots, 5).$$

Then the continued system for (3.2) and (3.3) takes the form

$$\begin{gathered} \frac{\partial T_k}{\partial z} + a_k \frac{\partial T_k}{\partial r} = L^k + L^k_\alpha T_\alpha + L^k_{\alpha\beta} T_\alpha T_\beta, \\ \frac{\partial t_k}{\partial z} = F^k + F^k_\alpha T_\alpha \quad (k, \alpha, \beta = 1, 2, \ldots, 5). \end{gathered} \tag{4.1}$$

Here

$$\begin{gathered} L^k = \frac{\partial f_k}{\partial r} \quad (k = 1, 2, \ldots, 5); \quad F^k = \frac{f_2 + (-1)^{k+1} f_1}{\Lambda} c \ (k = 1, 2); \quad \Lambda = \begin{cases} 2, & k = 1, \\ 4t_3, & k = 2; \end{cases} \\ F^3 = f_3; \quad F^4 = \frac{f_4 - f_5}{a_4 - a_5}; \quad F^5 = c; \\ L^k_\alpha = \frac{(-1)^{\alpha+k}}{4t_3} \left[\left(2 + \frac{\partial k_k}{\partial t_2} \right) c + (-1)^k \frac{\partial f_k}{\partial t_2} + 2f_3 \right] \quad (\alpha, k = 1, 2); \\ L^3_\alpha = (-1)^\alpha \left[\frac{\partial k_3}{\partial t_2} c + \frac{\partial f_3}{\partial t_2} \right] \quad (\alpha = 1, 2); \quad L^k_\alpha = \frac{(-1)^\alpha}{4t_3} \frac{\partial f_k}{\partial t_2} \quad (\alpha = 1, 2, \ k = 4, 5); \\ L^k_3 = 2 \left[\frac{1}{2} \frac{\partial f_k}{\partial t_3} + (-1)^{k+1} g + (-1)^{k+1} cd \right] \quad (k = 1, 2); \quad L^3_3 = \frac{\partial f_3}{\partial t_3}, \quad L^4_3 = L^5_3 = 0; \\ L^k_\alpha = \frac{a}{a_4 - a_5} \left[\left((-1)^{\alpha+k+1} 2f_3 + (-1)^{\alpha+1} \frac{\partial f_k}{\partial t_2} \right) d + \left((-1)^{k+1} k_3 + (-1)^{\alpha+k} \frac{\partial k_k}{\partial t_3} \right) g + \right. \\ \left. + (-1)^{\alpha-1} k_3 \frac{\partial f_k}{\partial t_3} + \frac{(-1)^{\alpha+k}}{a} \frac{\partial f_k}{\partial t_5} \right] \quad (k = 1, 2, \ \alpha = 4, 5); \quad a = \begin{cases} a_5, & k = 4, \\ a_4, & k = 5; \end{cases} \\ L^k_\alpha = \frac{a}{a_4 - a_5} \left[(-1)^{\alpha+1} \left(\frac{\partial f_k}{\partial t_2} d + \frac{\partial f_k}{\partial t_5} \right) + \frac{(-1)^\alpha}{a} (cd + g) \frac{\partial a_k}{\partial t_2} \right] \quad (\alpha, k = 4, 5); \\ L^3_\alpha = \frac{a}{a_4 - a_5} \left[(-1)^{\alpha+1} \frac{\partial f_3}{\partial t_2} d + (-1)^\alpha \frac{\partial k_3}{\partial t_2} g + (-1)^{\alpha+1} \frac{\partial f_3}{\partial t_3} k_3 + \frac{(-1)^\alpha}{a} \frac{\partial f_3}{\partial t_5} \right] \quad (\alpha = 4, 5); \\ F^1_\alpha = (-1)^\alpha \frac{a_1}{2}, \quad F^2_\alpha = (-1) \frac{a_\alpha}{4t_3} \quad (\alpha = 1, 2); \quad F^k_\alpha = 0 \ (\alpha = 1, 2, \ k = 3, 4, 5); \\ F^3_3 = -a_3; \quad F^k_3 = 0 \ (k = 1, 2, 4, 5); \\ F^1_\alpha = (-1)^\alpha \frac{k_2 - k_1}{2} \frac{a_4 a_5}{a_4 - a_5}, \quad F^2_\alpha = (-1)^\alpha d \frac{a_4 a_5}{a_4 - a_5}, \quad F^3_\alpha = (-1)^\alpha k_3 \frac{a_4 a_5}{a_4 - a_5}, \\ F^5_\alpha = (-1)^\alpha \frac{a_4 a_5}{a_4 - a_5} \quad (\alpha = 4, 5); \quad F^4_4 = \frac{2a_4 + a_5}{a_4 - a_5}; \quad F^4_5 = \frac{a_4 + 2a_5}{a_4 - a_5}; \\ L^k_{\alpha\beta} = L^k_{\alpha\beta} \left(t_3, a_i, k_i, \frac{\partial a_i}{\partial t_2}, \frac{\partial k_i}{\partial t_2} \right) \quad (i, k, \alpha, \beta = 1, 2, \ldots, 5); \\ c = \frac{a_4 f_5 - a_5 f_4}{a_4 - a_5}; \quad d = \frac{k_1 + k_2}{4t_3}; \quad g = \frac{f_2 - f_1}{4t_3}. \end{gathered} \tag{4.2}$$

The summation is over the Greek subscripts (it is missing over the subscript k).

Besides the system (4.1), we write the system of ordinary differential equations

$$dP/dz = L_0(N) + L_1(N)P + L_2(N)P^2, \ dN/dz = F_0(N) + F_1(N)P, \tag{4.3}$$

in which the coefficients of the right sides are determined as

$$L_0(N) = \max_{G_0(N)} \|\mathbf{L}\|, \quad \mathbf{L} = [L^1, L^2, \dots, L^5], \tag{4.4}$$

$$L_1(N) = \max_{G_0(N)} \|L_\alpha^k\|, \quad L_2(N) = \max_{G_0(N)} \max_{\beta=1,\dots,5} \|L_{\alpha\beta}^k\|,$$

$$F_0(N) = \max_{G_0(N)} \|\mathbf{F}\|, \quad \mathbf{F} = [F^1, F^2, \dots, F^5],$$

$$F_1(N) = \max_{G_0(N)} \|F_\alpha^k\|, \quad G_0(N) = \{a \leqslant r \leqslant b,\ 0 \leqslant z \leqslant z_0;\ \|\mathbf{t}\| \leqslant N\}.$$

Let N_0, P_0 denote the quantities

$$N_0 = \max_{a \leqslant r \leqslant b} \|\mathbf{S}_0(r) + \mathbf{U}_0(r)\|, \quad P_0 = \max_{a \leqslant r \leqslant b} \left\| \mathbf{l}_k^0 \cdot \frac{\partial(\mathbf{S}_0 + \mathbf{U}_0)}{\partial r} \right\|$$

and let us give the initial conditions

$$P(0) = P_0, \quad N(0) = N_0 \tag{4.5}$$

for the system (4.3). We call the system (4.3), (4.4) majorizing, and following [5] it can be shown that the functions $N(z)$, $P(z)$ which are a solution of the problem (4.3)-(4.5) and remain bounded here in the interval $0 \le z \le z_0$ majorize the growth of the solutions $\mathbf{t} = \mathbf{S} + \mathbf{U}$. The domain G_0 is constructed by means of the function $N(z)$.

Let us write down the continued system for equations (3.5), (3.6) (we write $(i + 1)$ in place of (i) in (3.5)) in the form

$$\frac{\partial T_k^{(i+1)}}{\partial z} + a_k^{(i+1)} \frac{\partial T_k^{(i+1)}}{\partial r} = L^k + \Lambda_\alpha^k T_\alpha^{(i)} + M_\alpha^k T_\alpha^{(i+1)} + L_{\alpha\beta}^k T_\alpha^{(i)} T_\beta^{(i+1)}, \tag{4.6}$$

$$\frac{\partial t_k^{(i+1)}}{\partial z} = F^k + F_\alpha^k T_\alpha^{(i+1)} \quad (k, \alpha, \beta = 1, 2, \dots, 5),$$

where

$$T_k^{(i+1)} = \mathbf{l}_k \cdot \frac{\partial(\mathbf{S} + \mathbf{U})^{(i+1)}}{\partial r}, \quad \mathbf{l}_k = \begin{cases} \mathbf{l}_k^{(i+1)} & (k = 1, 2, 3), \\ \mathbf{l}_k^{(i)} & (k = 4, 5), \end{cases}$$

and L^k, F^k, F_α^k, $L_{\alpha\beta}^k$ are determined in conformity with (4.2), except $\mathbf{t}^{(i)}$, $\mathbf{t}^{(i+1)}$ or $\mathbf{t}^{(i-1)}$, $\mathbf{t}^{(i)}$ should be substituted in place of $\mathbf{t}$. Only the coefficients Λ_α^k, M_α^k are different:

$$\Lambda_\alpha^k = \frac{a}{a_4^{(i-1)} - a_5^{(i-1)}} \left[(-1)^{\alpha+1} \left(\frac{\partial f_k^{(i)}}{\partial t_2} d^{(i)} + \frac{\partial f_k^{(i)}}{\partial t_5} \right) \right], \tag{4.7}$$

$$M_\alpha^k = \frac{a}{a_4^{(i-1)} - a_5^{(i-1)}} \left[\frac{(-1)^\alpha}{a} \left(c^{(i)} d^{(i)} + g^{(i)} \right) \frac{\partial a_k^{(i)}}{\partial t_2} \right]$$

$$(k, \alpha = 4, 5), \quad a = \begin{cases} a_5^{(i-1)}, & k = 4, \\ a_4^{(i-1)}, & k = 5. \end{cases}$$

Comparing the coefficients (4.7) with the analogous coefficients in (4.2), we obtain

$$L_\alpha^k(r, \mathbf{t}) = \Lambda_\alpha^k(r, \mathbf{t}, \mathbf{t}) + M_\alpha^k(r, \mathbf{t}, \mathbf{t}) \quad (k, \alpha = 4, 5), \tag{4.8}$$

$$L_{\alpha\beta}^k(r, \mathbf{t}) = L_{\alpha\beta}^k(r, \mathbf{t}, \mathbf{t}) \quad (k, \alpha, \beta = 1, 2, \dots, 5).$$

By analogy with (4.3), (4.4), we determine the majorizing system for the continued (4.6)

$$\frac{d\widetilde{P}}{dz} = L_0(\widetilde{N}) + K_1(\widetilde{N}) \widetilde{P} + K_2(\widetilde{N}) \widetilde{P}^2, \quad \frac{d\widetilde{N}}{dz} = F_0(\widetilde{N}) + F_1(\widetilde{N}) \widetilde{P}, \tag{4.9}$$

where L_0, F_0, F_1 are determined, as in (4.4), while

$$K_1(\tilde{N}) = K'(\tilde{N}) + K''(\tilde{N}), \quad K'(\tilde{N}) = \max_{G_0(\tilde{N})} \Lambda_\alpha^k(r, \mathbf{x}, \mathbf{t}),$$

$$\|\mathbf{t}\| \leqslant \tilde{N}, \quad \|\mathbf{x}\| \leqslant \tilde{N},$$

$$K''(\tilde{N}) = \max_{G_0(\tilde{N})} \max_{\beta=1,2,\ldots,5} \left\| L_{\alpha\beta}^k(r, \mathbf{x}, \mathbf{t}) \right\|,$$

$$\|\mathbf{t}\| \leqslant \tilde{N}, \quad \|\mathbf{x}\| \leqslant \tilde{N}.$$

Let us give the initial conditions

$$\tilde{P}(0) = P_0, \quad \tilde{N}(0) = N_0 \tag{4.10}$$

for (4.9). Comparing coefficients of the system (4.3), (4.9) and taking account of (4.8), we have

$$L_1(N) \leqslant K_1(N), \quad L_2(N) \leqslant K_2(N).$$

Therefore, the solution of the problem (4.9), (4.10) majorizes the growth of the solution of the majorizing problem (4.3)-(4.5)

$$N(z) \leqslant \tilde{N}(z), \quad P(z) \leqslant \tilde{P}(z).$$

Therefore, if the domain G_1 is constructed by means of the function $\tilde{N}(z)$ as G_0 is constructed by means of $N(z)$ [5], then $G_1 \subseteq G_0$.

Let us assume that all the successive approximations $(S + U)^{(j)}$ $(j = 1, 2, \ldots, i)$ satisfy the inequalities

$$\|\mathbf{t}^{(j)}\| \leqslant \tilde{N}(z), \quad \|T^{(j)}\| \leqslant \tilde{P}(z), \tag{4.11}$$

and let us show that (4.11) holds even for the $(i + 1)$-th approximation. Let us use the notation

$$t_{i+1}(z) = \sup_r \|\mathbf{t}^{(i+1)}\|, \quad T_{i+1}(z) = \sup_r \|T^{(i+1)}\|;$$

then we will have from the system (4.6)

$$\frac{dT_{i+1}}{dz} \leqslant L_0(\tilde{N}) + K'(\tilde{N})\tilde{P} + K''(\tilde{N})T_{i+1} + K_2(\tilde{N})\tilde{P}T_{i+1},$$

$$\frac{dt_{i+1}}{dz} \leqslant F_0(\tilde{N}) + F_1(\tilde{N})\tilde{P},$$

so that evidently

$$t_{i+1}(z) \leqslant \tilde{N}(z), \quad T_{i+1}(z) \leqslant \tilde{P}(z).$$

Since the initial approximation can be selected to satisfy (4.11), then it is thereby proved that all the approximations $(S + U)^{(i)}$ satisfy (4.11), and therefore, the domain G_1 exists and belongs to all the domains $G^{(i)}$ and G_0.

5. Uniform Convergence of the Successive Approximations in G_1. If the residual

$$\rho_k^{(i+1)} = \mathrm{l}_k^{(i)} \cdot (\mathrm{U}^{(i+1)} - \mathrm{U}^{(i)}) \quad (k = 4, 5), \tag{5.1}$$

is introduced, then following [8] it can be shown that for small $0 \leqslant z \leqslant z_0$ the estimate

$$R_{i+1}(z) = \max_{\tau, r \in G_1, \tau \leqslant r} \|\rho^{(i+1)}\| \leqslant (\exp(qz) - 1)\|\mathrm{S}^{(i)} - \mathrm{S}^{(i-1)}\|, \quad q = \mathrm{const} \tag{5.2}$$

holds. Furthermore, by using (5.2), we prove the uniform convergence of the successive approximations $\{S^{(i)}\}$. To do this, we define analogously to (5.1)

$$\delta_k^i = \mathbf{l}_k^{(i)} \cdot (\mathrm{S}^{(i)} - \mathrm{S}^{(i-1)}) \quad (k = 1, 2, 3) \tag{5.3}$$

and we write the system (3.5) corresponding to the (i − 1)-th approximation

$$l_k^{(i-1)}\cdot\left[\frac{\partial S^{(i-1)}}{\partial z}+a_k^{(i-1)}\cdot\frac{\partial S^{(i-1)}}{\partial r}\right]=f_k^{(i-1)}-l_k^{(i-1)}\cdot\left[\frac{\partial U^{(i-1)}}{\partial z}+a_k^{(i-1)}\frac{\partial U^{(i-1)}}{\partial r}\right]. \tag{5.4}$$

Using the theorem on finite increments, we have, for instance

$$f_k^{(i)}-f_k^{(i-1)}=\int_0^1\frac{\partial f_k}{\partial t_\beta}\left(\mathbf{t}^{(i-1)}+\lambda\left(\mathbf{t}^{(i)}-\mathbf{t}^{(i-1)}\right)\right)d\lambda\left(t_\beta^{(i)}-t_\beta^{(i-1)}\right). \tag{5.5}$$

We now subtract (5.4) from (3.5), respectively, by using here the relationship (5.5), and similar ones for $l_k^{(i)}-l_k^{(i-1)}$, $a_k^{(i)}l_k^{(i)}-a_k^{(i-1)}l_k^{(i-1)}$, and also by taking account of the possibility of the resolution of (5.1), (5.3) with respect to $u_k^{(i+1)}-u_k^{(i)}$ (k = 4, 5), $S_k^{(i)}-S_k^{(i-1)}$ (k = 1, 2, 3) we arrive at a linear system for

$$\frac{\partial\delta_k^{(i)}}{\partial z}+a_k^{(i)}\frac{\partial\delta_k^{(i)}}{\partial r}=\Pi_k^\beta\delta_\beta^{(i)}+X_k^\alpha\rho_\alpha^{(i)}\quad(k,\ \beta=1,2,3,\ \alpha=4,5). \tag{5.6}$$

Here Π_k^β, X_k^α are defined in terms of the functions f_k, l_k, U, S, and their first derivatives. Integrating (5.6) along the characteristics in G_1, we obtain for each point of G_1

$$|\delta_k^{(i)}|\leqslant\int_0^z|\Pi_k^\beta\delta_\beta^{(i)}+X_k^\alpha\rho_\alpha^{(i)}|\,d\tau,$$

where by virtue of (4.11), the following inequalities hold

$$\|\Pi_k^\beta\|\leqslant E,\quad\|X_k^\alpha\|\leqslant E,\quad E=\text{const}.$$

Then if we introduce

$$D_i(z)=\max_{\tau,r\in G_1,\tau\leqslant z}\|\delta^{(i)}\|$$

and use the estimating inequality (5.2) written for $R_i(z)$ we obtain

$$D_i(z)\leqslant E\int_0^z[D_{i-1}(\tau)+D_i(\tau)]\,d\tau,\quad D_i(z)\leqslant C\int_0^z D_{i-1}(\tau)\,d\tau$$

or

$$D_i(z)\leqslant\text{const}\frac{(Cz)^{(i-1)}}{(i-1)!},\quad C=\text{const}, \tag{5.7}$$

which indeed proves the uniform convergence of the successive approximations $\{S^{(i)}\}$ in G_1. It is evident that the uniform convergence of the sequence $\{U^{(i)}\}$ in G_1 also follows from (5.2) and (5.7). Finally, considering the continued system (4.6) and using the continuous dependence of the solution of the Cauchy problem on the initial data, we arrive at the uniform convergence of the sequences $\{T^{(i)}\}$, $\{\partial t^{(i)}/\partial z\}$, and therefore, of $\{\partial t^{(i)}/\partial r\}$ as well. By means of a known theorem of analysis, this means that the vector functions $S=\lim S^{(i)}$, $U=\lim U^{(i)}$ $(i\to\infty)$ are continuously differentiable in G_1. Passing to the limit in (3.5), (3.6), we conclude that $\mathbf{t}=S+U$ is a solution of the problem (3.2)-(3.4). Since the proof of the convergence of the method of solving the Cauchy problem considered for the system (3.2) and (3.3) was based on the characteristic approach, then it will carry over to the case of the characteristic and mixed problems without substantial changes.

Therefore, the proposed hyperbolic regularization of the ideal plasticity equations and the iteration method of solving boundary-value problems for the system (2.7), (2.8) permit solving axisymmetric problems with Mises and Tresk conditions in a rigidly plastic formulation for any arbitrarily small $\varepsilon=\varepsilon_0>0$. The rate of mechanical energy dissipation will be nonnegative everywhere in the flow domain. It is convenient to use the solution of (2.3) on a slip line mesh of the complete plasticity state [2, 9] as the first approximation of the velocity field. The condition of embedding of the domains of determinacy of each of the subsequent approximations, starting with the first, is here satisfied, which turns out to be essential for the numerical realization of the method of characteristics in specific problems since they are nonlinear.

In conclusion, we note the following.

1) It can be shown that the finite-difference equations corresponding to the system (2.8), (2.9) satisfy the conditions of the regularizing operator [10], for which it is sufficient to linearize the system (2.1) by using iteration of the previous step in determining the matrices A and B. Here A and B are determined approximately and are poorly specified, which allows introduction of the small parameter ε which is included in the scheme to construct the regularizing operator [10] (convergence in ε is not considered in this paper).

2) The presence of the parameter γ in (2.7) permits obtaining a qualitative and quantitative description of the phenomenon of formation of dead zones, domes, pulsations in conical outflow, drawing, pressing, and insertion problems. Assuming γ a random function with the values ± 1 depending on the geometry of the flow process, the boundary conditions, and other factors, we arrive at a flow with domains in which the multiplicity of the characteristics α, β changes. The active flow surfaces along which the third of the relationships (2.8) is satisfied can thereby be alternately α and β surfaces. The alternation of such domains is generally random in nature, by which the above-mentioned phenomena are perhaps explained.

LITERATURE CITED

1. R. Hill, Mathematical Theory of Plasticity, Oxford University Press (1950).
2. D. D. Ivlev, Theory of Ideal Plasticity [in Russian], Nauka, Moscow (1966).
3. M. Sh. Shtein, "On certain exact solutions of the ideal plasticity equations in the axial symmetry case," Prikl. Mekh., 19, No. 10 (1983).
4. R. Courant, Partial Differential Equations [Russian translation], Mir, Moscow (1964).
5. B. L. Rozhdestvenskii and N. N. Yanenko, Systems of Quasilinear Equations and Their Application to Gas Dynamics [in Russian], Nauka, Moscow (1968).
6. P. S. Symond, "On the general equations of problems of axial symmetry in the theory of plasticity," Q. J. Appl. Math., 6, 448 (1949).
7. A. F. Revuzhenko and E. I. Shemyakin, "Certain formulations of plasticity boundary-value problems," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1979).
8. M. Sh. Shtein, "On an approximate method of solving ideal plasticity theory equations," Prikl. Mat. Mekh., 38, No. 4 (1974).
9. M. Sh. Shtein, "Stress and strain state of a half-space in the neighborhood of the end-face of a circular cylindrical recess," Fiz.-Tekh. Prob. Razrabot. Poluzn. Iskop., No. 4 (1975).
10. A. N. Tikhonov and V. Ya. Arsenin, Methods of Solving Incorrect Problems [in Russian], Nauka, Moscow (1979).